Dirac Equation with Spin Symmetry for the Modified Pöschl-Teller Potential in D-dimensions

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Abstract We present solutions of the Dirac equation with spin symmetry for vector and scalar modified Pöschl-Teller potential within framework of an approximation of the centrifugal term. The relativistic energy spectrum is obtained using the Nikiforov-Uvarov method and the two-component spinor wavefunctions are obtain are in terms of the Jacobi polynomials. It is found that there exist only positive-energy states for bound states under spin symmetry, and the energy levels increase with the dimension and the potential range parameter α.

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Keywords Dirac equation, Modified Pöschl-Teller potential, Spin symmetry and Nikiforov-Uvarov method.

Solutions to relativistic equations play a very important in many aspects of mordern physics. In particular, the Dirac equation which describe the motion of a spin-$\frac{1}{2}$ particle has been used in solving many problems of nuclear aand high-energy physics. The spin symmetry arises if the magnitude of the attractive Lorentz scalar potential $S(r)$ and the replusive vector potential $V(r)$ are nearly equal, i.e $S(r) \sim V(r)$, in the nuclei, while the pseudo-spin symmetry occur when $S(r) \sim -V(r)$ [1-3]. The case of the exact spin and pseudo-spin symmetries has been shown to correspond to the $SU(2)$ symmetries of the Dirac Hamiltonian [3]. The spin symmetry is relevant for mesons [4] and the pseudo-spin symmetry is used to explain deformed nuclei [5], super-deformation [6] and to establish an effective nuclear shell-model scheme [7-9]. Also, various potentials such as the Morse potential [10-12], Wood-saxon potential [13], Coulomb and Hartmann potentials [14], Eckart potential [15, 16], Pöschl-Teller potential [17, 18] and the harmonic potential [19, 20] have been studied within the frame work of the spin and pseudospin symmetries.

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Moreover, with the interest in higher dimensional theory, the multi-dimensional quantum mechanical equations-relativistic and nonrelativistic- have been solved with various physical potentials. To mention a few, the D-dimensional Schrödinger has been studied with the Coulomb-like potential [21], pseudoharmonic potential [22], Hulthén potential [23] and Pöschl-Teller potential [24]. In addition, the D-dimensional relativistic Klein-Gordon and Dirac equations have been studied with many exactly solvable models [25-30]. However, some physical potentials can only be solve exactly for the s-states. Unfortunately, the modified Pöschl-Teller potential is one of these potentials. For instance, a recent work [31] has presented the s-wave solutions of the Dirac equation with the Pöschl-Teller potential under the conditions of the exact spin symmetry and pseudospin symmetry. However, in order to extend the solutions of the modified Pöschl-Teller potential to any $\ell \neq 0$ state, some recent studies [17, 24] have used a hyperbolic approximation for the centrifugal term to obtain the non-relativistic solutions of the modified Pöschl-Teller potential. In light of this, the present Letter intends to extend the discussions on the relativistic Pöschl-Teller potential to D-dimensions by presenting the bound-state solutions of the D-dimensions Dirac equation with spin symmetry for Lorentz vector and scalar modified Pöschl-Teller potential using the Nikiforov-Uvarov method [32].

The D-dimensional Dirac equation with a scalar potential $V_s(r)$ and a vector potential $V_v(r)$ and mass μ can be written in natural units $\hbar = c = 1$ as [27, 33, 34]

$$H\Psi(r) = E_{n_r\kappa}\Psi(r) \quad \text{where} \quad H = \sum_{j=1}^{D} \hat{\alpha}_j p_j + \hat{\beta}[\mu + V_s(r)] + V_v(r) \tag{1}$$

where $E_{n_r\kappa}$ is the relativistic energy, $\{\hat{\alpha}_j\}$ and $\hat{\beta}$ are Dirac matrices, which satisfy anti-commutation relations

$$\begin{aligned} \hat{\alpha}_j\hat{\alpha}_k + \hat{\alpha}_k\hat{\alpha}_j &= 2\delta_{jk}\mathbf{1} \\ \hat{\alpha}_j\hat{\beta} + \hat{\beta}\hat{\alpha}_j &= 0 \\ \hat{\alpha}_j^2 = \hat{\beta}^2 &= \mathbf{1} \end{aligned} \tag{2}$$

and

$$p_j = -i\partial_j = -i\frac{\partial}{\partial x_j} \quad 1 \leqslant j \leqslant D. \tag{3}$$

The orbital angular momentum operators L_{jk}, the spinor opertaors S_{jk} and the total angular momentum operators J_{jk} can be defined as follows:

$$L_{jk} = -L_{jk} = ix_j\frac{\partial}{\partial x_k} - ix_k\frac{\partial}{\partial x_j}, \quad S_{jk} = -S_{kj} = i\hat{\alpha}_j\hat{\alpha}_k/2, \quad J_{jk} = L_{jk} + S_{jk}.$$

$$L^2 = \sum_{j<k}^{D} L_{jk}^2, \quad S^2 = \sum_{j<k}^{D} S_{jk}^2, \quad J^2 = \sum_{j<k}^{D} J_{jk}^2, \quad 1 \leqslant j < k \leqslant D. \tag{4}$$

For a spherically symmetric potential, total angular momentum operator J_{jk} and the spin-orbit operator $\hat{K} = -\hat{\beta}(J^2 - L^2 - S^2 + (D-1)/2)$ commutate with the Dirac Hamiltonian. For a given total angular momentum j, the eigenvalues of $\hat{K}$ are $\kappa = \pm(j + (D-2)/2)$; $\kappa = -(j + (D-2)/2)$ for aligned spin $j = \ell + \frac{1}{2}$ and $\kappa = (j + (D-2)/2)$ for unaligned spin $j = \ell - \frac{1}{2}$.

Thus, we can introduce the hyperspherical coordinates [35]

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_\alpha &= r\sin\theta_1 \ldots \sin\theta_{\alpha-1}\cos\phi, \quad 2 \leqslant \alpha \leqslant D-1 \\ x_D &= r\sin\theta_1 \ldots \sin\theta_{D-2}\sin\phi, \end{aligned} \tag{5}$$

where the volume element of the configuration space is given as

$$\prod_{j=1}^{D} dx_j = r^{D-1} dr d\Omega \qquad d\Omega = \prod_{j=1}^{D-1} (\sin\theta_j)^{j-1} d\theta_j \tag{6}$$

with $0 \leqslant r < \infty$, $0 \leqslant \theta_k \leqslant \pi$, $k = 1, 2, \ldots D-2$, $0 \leqslant \phi \leqslant 2\pi$, such that the spinor wavefunctions can be classified according to the hyperradial quantum number n_r and the spin-orbit quantum number κ and can be written using the Pauli-Dirac representation

$$\Psi_{n_r\kappa}(r, \Omega_D) = r^{-\frac{D-1}{2}} \begin{pmatrix} F_{n_r\kappa}(r) Y_{jm}^{\ell}(\Omega_D) \\ \\ iG_{n_r\kappa}(r) Y_{jm}^{\tilde{\ell}}(\Omega_D) \end{pmatrix} \tag{7}$$

where $F_{n_r\kappa}(r)$ and $G_{n_r\kappa}(r)$ are the radial wave function of the upper- and the lower-spinor components respectively, $Y_{jm}^{\ell}(\Omega_D)$ and $Y_{jm}^{\tilde{\ell}}(\Omega_D)$ are the hyperspherical harmonic functions coupled with the total angular momentum j. The orbital and the pseudo-orbital angular momentum quantum numbers for spin symmetry ℓ and and pseudospin symmetry $\tilde{\ell}$ refer to the upper- and lower-component respectively.

Substituting Eq. (7) into Eq. (1), and seperating the variables we obtain the following coupled radial Dirac equation for the spinor components:

$$\left(\frac{d}{dr} + \frac{\kappa}{r}\right) F_{n_r\kappa}(r) = [\mu + E_{n_r\kappa} - \Delta(r)] G_{n_r\kappa}(r) \tag{8}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r}\right) G_{n_r\kappa}(r) = [\mu - E_{n_r\kappa} + \Sigma(r)] F_{n_r\kappa}(r) \tag{9}$$

where $\Delta(r) = V_v(r) - V_s(r)$, $\Sigma(r) = V_v(r) + V_s(r)$ and $\kappa = \pm(2\ell + D - 1)/2$. Further details of the derivation can be obtain from refs [36-38]. Using Eq. (8) as the upper component and substituting into Eq. (9), we obtain the follwoing second order differential equations

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - [\mu + E_{n_r\kappa} - \Delta(r)][\mu - E_{n_r\kappa} + \Sigma(r)] + \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr} + \frac{\kappa}{r}\right)}{[\mu(r) + E_{n_r\kappa} - \Delta(r)]}\right] F_{n_r\kappa}(r) = 0 \tag{10}$$

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - [\mu + E_{n_r\kappa} - \Delta(r)][\mu - E_{n_r\kappa} + \Sigma(r)] - \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr} - \frac{\kappa}{r}\right)}{[\mu(r) - E_{n_r\kappa} + \Sigma(r)]}\right] G_{n_r\kappa}(r) = 0 \tag{11}$$

We note that the energy eigenvalues in these equation depend on the angular momentum quantum number ℓ and dimension D. However, to solve these equations, we shall use an approximation for the centrifugal barrier and obtain the solutions using the Nikiforov-Uvarov method.

Next, we give a brief description of the conventional Nikiforov-Uvarov method. A more detailed description of the method can be obtained the following reference [32]. With an appropriate transformation $s = s(r)$,the one dimensional Schr$\ddot{o}$dinger equation can be reduced to a generalized equation of hypergeometric type which can be written as follows:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{12}$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second-degree, and $\tilde{\tau}(s)$is at most a first-order polynomial. To find particular solution of Eq. (12) by separation of variables, if one deals with

$$\psi(s) = \phi(s) y_{n_r}(s), \tag{13}$$

Eq. (12) becomes

$$\sigma(s)y''_{n_r} + \tau(s)y'_{n_r} + \lambda y_{n_r} = 0 \tag{14}$$

where

$$\sigma(s) = \pi(s)\frac{\phi(s)}{\phi'(s)} \tag{15}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \tau'(s) < 0, \tag{16}$$

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + t\sigma}, \tag{17}$$

and

$$\lambda = t + \pi'(s). \tag{18}$$

The polynomial $\tau(s)$ with the parameter s and prime factors show the differentials at first degree be negative. However, determination of parameter t is the essential point in the calculation of $\pi(s)$. It is simply defined by setting the discriminate of the square root to zero [32]. Therefore, one gets a general quadratic equation for t. The values of t can be used for calculation of energy eigenvalues using the following equation

$$\lambda = t + \pi'(s) = -n_r\tau'(s) - \frac{n_r(n_r-1)}{2}\sigma''(s). \tag{19}$$

Furthermore, the other part $y_{n_r}(s)$ of the wave function in Eq. (12) is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation:

$$y_{n_r}(s) = \frac{B_{n_r}}{\rho(s)}\frac{d^{n_r}}{ds^{n_r}}[\sigma^{n_r}(s)\rho(s)] \tag{20}$$

where B_{n_r} is a normalizing constant and the weight function $\rho(s)$ must satisfy the condition [32]

$$(\sigma\rho)' = \tau\rho. \tag{21}$$

The Lorentz vector $V_v(r)$ and scalar $V_s(r)$ modified Pöschl-Teller potential can be defined as follows [24, 39-41]

$$V_v(r) = -\frac{V_0}{\cosh^2(\alpha r)} \quad \text{and} \quad V_s(r) = -\frac{S_0}{\cosh^2(\alpha r)} \tag{22}$$

where α is related to the range of the potential and V_0 and S_0 are the depths of the vector and scalar potentials respectively. Moreover, we can approximate the centrifugal terms as follows [17, 24]

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\sinh^2(\alpha r)}. \tag{23}$$

Substituting Eqs. (22) and (23) into Eqs. (10) and (11), we have

$$\left[\frac{d^2}{dr^2} - \frac{\alpha^2\kappa(\kappa+1)}{\sinh^2(\alpha r)} - [\mu + E_{n_r\kappa} - \Delta(r)][\mu - E_{n_r\kappa} + \Sigma(r)] + \frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr} + \frac{\kappa}{r}\right)}{[\mu(r) + E_{n_r\kappa} - \Delta(r)]}\right] F_{n_r\kappa}(r) = 0 \tag{24}$$

$$\left[\frac{d^2}{dr^2} - \frac{\alpha^2\kappa(\kappa-1)}{\sinh^2(\alpha r)} - [\mu + E_{n_r\kappa} - \Delta(r)][\mu - E_{n_r\kappa} + \Sigma(r)] - \frac{\frac{d\Sigma(r)}{dr}\left(\frac{d}{dr} - \frac{\kappa}{r}\right)}{[\mu(r) - E_{n_r\kappa} + \Sigma(r)]}\right] G_{n_r\kappa}(r) = 0 \tag{25}$$

where

$$\Delta(r) = \frac{S_0 - V_0}{\cosh^2(\alpha r)} \quad \text{and} \quad \Sigma(r) = \frac{-(V_0 + S_0)}{\cosh^2(\alpha r)}. \tag{26}$$

For the case of spin symmetry, $V_v(r) \sim V_s(r)$, i.e. $\Delta(r) = V_v(r) - V_s(r) = C_1$ (a constant), which implies that $\frac{d\Delta(r)}{dr} = 0$. Thus, putting this into Eq. (24), we have

$$\left[\frac{d^2}{dr^2} - \frac{\alpha^2\kappa(\kappa+1)}{\sinh^2(\alpha r)} - (\mu - E_{n_r\kappa})(\mu + E_{n_r\kappa} - C_1) + \frac{(V_0 + S_0)(E_{n_r\kappa} + \mu - C_1)}{\cosh^2(\alpha r)}\right] F_{n_r\kappa}(r) = 0. \tag{27}$$

If we take the transformation $s = \tanh^2(\alpha r)$, Eq. (27) becomes

$$F''_{n_r\kappa}(s) + \frac{1-3s}{2s(1-s)} F'_{n_r\kappa}(s) + \frac{1}{4s^2(1-s)^2}[-\delta s^2 + (\delta + \gamma - \epsilon^2)s - \gamma] F_{n_r\kappa} = 0 \tag{28}$$

where

$$\epsilon^2 = \frac{(\mu - E_{n_r\kappa})(\mu + E_{n_r\kappa} - C_1)}{\alpha^2}, \quad \delta = \frac{(V_0 + S_0)(E_{n_r\kappa} + \mu - C_1)}{\alpha^2} \quad \text{and} \quad \gamma = \kappa(\kappa+1). \tag{29}$$

Comparing Eqs. (28) and (12) we can define the following

$$\tilde{\tau}(s) = 1 - 3s, \quad \sigma(s) = 2s(1-s) \quad \text{and} \quad \tilde{\sigma}(s) = -\delta s^2 + (\gamma + \delta - \epsilon^2)s - \gamma \tag{30}$$

Inserting these into Eq. (17), we have the following function

$$\pi(s) = \frac{1-s}{2} \pm \frac{1}{2}\sqrt{(1 + 4\delta - 8t)s^2 + (8t - 4(\gamma + \delta - \epsilon^2) - 2)s + 4\gamma + 1} \tag{31}$$

The constant parameter t can be found by the condition that the discriminant of the expression under the square root has a double root, i.e., its discriminant is zero. Thus the possible value function for each value of t is given as

$$\pi(s) = \frac{1-s}{2} \pm \begin{cases} \frac{1}{2}\left[\left(-2\epsilon + \sqrt{1+4\gamma}\right)s - \sqrt{1+4\gamma}\right] & \text{for} \quad t = -\frac{1}{2}(\gamma - \delta + \epsilon^2) + \frac{1}{2}\epsilon\sqrt{1+4\gamma} \\ \frac{1}{2}\left[\left(2\epsilon + \sqrt{1+4\gamma}\right)s - \sqrt{1+4\gamma}\right] & \text{for} \quad t = -\frac{1}{2}(\gamma - \delta + \epsilon^2) - \frac{1}{2}\epsilon\sqrt{1+4\gamma} \end{cases} \tag{32}$$

By Nikiforov-Uvarov method, we made an appropriate choice of the function $\pi(s) = \frac{1-s}{2} - \frac{1}{2}\left[\left(2\epsilon + \sqrt{1+4\gamma}\right)s - \sqrt{1+4\gamma}\right]$ such that by Eq. (19), we can obtain the eigenvalue equation to be

$$-\frac{1}{2}(\gamma - \delta + \epsilon^2) - \frac{1}{2}\epsilon\sqrt{1+4\gamma} - \frac{1}{2}(2\epsilon + \sqrt{1+4\gamma}) - \frac{1}{2} = n_r[4 + 2\epsilon + \sqrt{1+4\gamma}] + 2n_r(n_r - 1) \tag{33}$$

Eq. (33) can be written in the powers of ϵ as follows

$$\epsilon^2 + \epsilon\left[2(2n_r + 1) + \sqrt{1+4\gamma}\right] + (\gamma - \delta) + \left[(1 + 2n_r) + \sqrt{1+4\gamma}\right] = 0, \tag{34}$$

such that we can obtain

$$-\epsilon^2 = -\frac{1}{4}\left[-2(2n_r + 1) - \sqrt{1+4\gamma} + \sqrt{1+4\delta}\right]^2, \tag{35}$$

from which we can obtain a rather complicated transcendental energy equation:

$$(\mu - E_{n_r\kappa})(\mu + E_{n_r\kappa} - C_1) = \frac{\alpha^2}{4}\left[2(2n_r + 1) + (2\kappa + 1) - \frac{1}{\alpha}\sqrt{\alpha^2 + 4(V_0 + S_0)(E_{n_r\kappa} + \mu - C_1)}\right]^2 \tag{36}$$

If we define a principal quantum number $n = 2n_r + \ell + 1$, Eq. (36) becomes

$$(\mu - E_n)(\mu + E_n - C_1) = \frac{\alpha^2}{4}\left[2n + D - \frac{1}{\alpha}\sqrt{\alpha^2 + 4(V_0 + S_0)(E_n + \mu - C_1)}\right]^2 \tag{37}$$

where we have chosen $\kappa = (2\ell + D - 1)/2$ and $n = 1, 2, 3, \ldots$ Some numerical values of the energy levels $E(\alpha, n, D)$ for some dimensions and exited states are given in Table 1.

We now obtain the spinor components of the wavefunction for the spin symmetry case using the Nikiforov-Uvarov method. By substituting $\pi(s)$ and $\sigma(s)$ into Eq. (15), and solving the first order differential equation to have

$$\phi(s) = s^{(\kappa+1)/2}(1 - s)^{\epsilon/2}. \tag{38}$$

Also using Eq. (18), the weight function $\rho(s)$ can be obtained as

$$\rho(s) = \frac{1}{2}s^{(2\kappa-1)/2}(1 - s)^{\epsilon} \tag{39}$$

Substituting Eq. (39) into the Rodrigues relation (20), we have

$$y_{n_r}(s) = B_{n_r}s^{-(2\kappa-1)/2}(1 - s)^{-\epsilon}\frac{d^{n_r}}{ds^{n_r}}\left[s^{n_r+(2\kappa-1)/2}(1 - s)^{n_r+\epsilon}\right]. \tag{40}$$

Therefore, we can write the upper component $F_{n_r\kappa}(s)$ as

$$F_{n_r\kappa}(s) = C_{n_r}s^{(\kappa+1)/2}(1 - s)^{\epsilon/2}P_{n_r}^{((2\kappa-1)/2,\ \epsilon)}(1 - 2s) \tag{41}$$

where C_{n_r} is the normalization constant, and we have used the definition of the Jacobi polynomials [42], given as

$$P_n^{(a,\ b)}(s) = \frac{(-1)^n}{n!2^n(1 - s)^a(1 + s)^b}\frac{d^n}{ds^n}\left[(1 - s)^{a+n}(1 + s)^{b+n}\right]. \tag{42}$$

The lower-component can be obtain as follows using Eq. (8)

$$G_{n_r\kappa}(s) = A_1(s)P_{n_r}^{((2\kappa-1)/2,\ \epsilon)}(1 - 2s) + A_2(s)P_{n_r-1}^{(2\kappa+1)/2,\ \epsilon/2+1)}(1 - 2s) \tag{43}$$

where

$$A_1(s) = \frac{C_{n_r}\alpha s^{\kappa/2}(1 - s)^{\epsilon/2}\left[\left(\frac{\kappa+1}{2}\right)(1 - s) - \frac{\epsilon}{2}s\right] + C_{n_r}\frac{\alpha\kappa}{\tanh^{-1}(\sqrt{s})}}{\mu + E_{n_r\kappa} - C_1}$$

and

$$A_2(s) = \frac{D_{n_r}\alpha s^{(\kappa+2)/2}(1 - s)^{(\epsilon+2)/2}}{\mu + E_{n_r\kappa} - C_1} \tag{44}$$

with constant D_{n_r} defined by

$$D_{n_r} = \frac{2\epsilon + 2n_r + 2\kappa + 1}{4} \times C_{n_r} \tag{45}$$

. Moreover, to compute the normalization constant C_{n_r}, it is easy to show that

$$\int_0^{\infty}\left|r^{\frac{-(D-1)}{2}}F_{n_r\kappa}(r)\right|^2 r^{D-1}dr = \int_0^{\infty}|F_{n_r\kappa}(r)|^2dr = \int_0^1 |F_{n_r\kappa}(s)|^2\frac{ds}{2\alpha\sqrt{s}(1 - s)} = 1 \tag{46}$$

where we have also used the substitution $s = \tanh^2(\alpha r)$. Putting Eq. (41) into Eq. (46) and using the following definition of the Jacobi polynomial [42]

$$P_n^{(a,b)}(s) = \frac{\Gamma(n + a + 1)}{n!\Gamma(1 + a)}\ {}_2F_1\left(-n, a + b + n + 1; 1 + a; \frac{1 - s}{2}\right), \tag{47}$$

we arrived at

$$C_{n_r}^2 N_{n_r} \int_0^1 s^{\kappa+\frac{1}{2}}(1-s)^{\epsilon-1}[\,{}_2F_1\,(-n_r, \kappa+\epsilon+n_r+1/2;\;\kappa+1/2;\;s)]^2 ds = \alpha \tag{48}$$

where $N_{n_r} = \frac{1}{2}\left[\frac{\Gamma(n_r+\kappa+1/2)}{n_r!\Gamma(\kappa+1/2)}\right]^2$ and ${}_2F_1$ is the hypergeometric function. Using the following series representation of the hypergeometric function

$${}_pF_q(a_1, ..., a_p; c_1, ..., c_q; s) = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(c_1)_n...(c_q)_n}\frac{s^n}{n!} \tag{49}$$

we have

$$C_{n_r}^2 N_{n_r} \sum_{i=0}^{n_r}\sum_{j=0}^{n_r} \frac{(-n_r)_i(\kappa+\epsilon+n_r+1/2)_i}{(\kappa+1/2)_i i!}\frac{(-n_r)_j(\kappa+\epsilon+n_r+1/2)_j}{(\kappa+1/2)_j j!}\int_0^1 s^{\kappa+i+j+\frac{1}{2}}(1-s)^{\epsilon-1}ds = \alpha. \tag{50}$$

Hence, by the definition of the Beta function, Eq. (43) becomes

$$C_{n_r}^2 N_{n_r} \sum_{i=0}^{n_r}\sum_{j=0}^{n_r} \frac{(-n_r)_i(\kappa+\epsilon+n_r+1/2)_i}{\kappa+1/2)_i i!}\frac{(-n_r)_j(\kappa+\epsilon+n_r+1/2)_j}{(\kappa+1/2)_j j!} B\left(\kappa+i+j+\frac{3}{2}, \epsilon\right) = \alpha. \tag{51}$$

Using the relations $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and the Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, Eq. (51) can be written as

$$C_{n_r}^2 N_{n_r} \sum_{i=0}^{n_r} \frac{(-n_r)_k(\kappa+\epsilon+n_r+1/2)_i(\kappa+\frac{3}{2})_i}{(\epsilon+\kappa+\frac{3}{2})_k(\kappa+1/2)_i k!} \sum_{j=0}^{n_r} \frac{(-n_r)_j(\kappa+\epsilon+n_r+1/2)_j(\kappa+i+\frac{3}{2})_j}{(\epsilon+\kappa+i+\frac{3}{2})_j(\kappa+1/2)_j j!} = \frac{\alpha}{B(\kappa+\frac{3}{2}, \epsilon)} \tag{52}$$

Lastly, Eq. (52) can be used to compute the normalization constants for $n_r = 0, 1, 2, ...$ In particular for the ground state, i.e $n_r = 0$, we have

$$C_0 = \sqrt{\frac{2\alpha}{B\left(\kappa+\frac{3}{2}, \epsilon\right)}}. \tag{53}$$

In conclusion, the solutions of the Dirac equation with spin symmetry for the modified Pöschl-Teller potential has been extended to a multi-dimensional case. The energy levels and the spinor-components of the wavefunction were obtained using the Nikiforov-Uvarov method. We also obtain the normalization constants in form of the hypergeometric series. Numerical results show that there are only positive-energy states for bound states with spin symmetry. Also, the energy levels increase with the dimension and the potential range parameter α. Moreover, the existence of the degenerate states between $E(\alpha, n+1, D)$ and $E(\alpha, n, D+2)$ indicate that the energy levels can be completely determined using the ground state.

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Table 1: The bound-state energy levels E_n are shown in the case of spin symmetry. The numerical results show that the energy levels increase with both the dimensions D and the range parameter α.

$E(\alpha, n, D)$					
$C_1 = V_0 = S_0 = \mu = 1$					
D	n	$\alpha = 0.0001$	$\alpha = 0.001$	$\alpha = 0.005$	$\alpha = 0.01$
3	1	4.0032×10^{-8}	4.0326×10^{-6}	1.0442×10^{-4}	4.4016×10^{-4}
	2	9.0108×10^{-8}	9.1118×10^{-6}	2.4161×10^{-4}	1.1130×10^{-3}
	3	1.6026×10^{-7}	1.6271×10^{-5}	4.4842×10^{-4}	—
	4	2.5050×10^{-7}	2.5544×10^{-5}	—	—
	5	3.6087×10^{-7}	3.6973×10^{-5}	—	—
4	1	6.2562×10^{-8}	6.3142×10^{-6}	1.6530×10^{-4}	7.1490×10^{-4}
	2	1.2267×10^{-7}	1.2429×10^{-5}	3.3488×10^{-4}	—
	3	2.0287×10^{-7}	2.0641×10^{-5}	6.0121×10^{-4}	—
	4	3.0317×10^{-7}	3.0955×10^{-5}	—	—
	5	4.2361×10^{-7}	4.3513×10^{-5}	—	—
5	1	9.0108×10^{-8}	9.1118×10^{-6}	2.4161×10^{-4}	1.1131×10^{-3}
	2	1.6026×10^{-7}	1.6270×10^{-5}	4.4842×10^{-4}	—
	3	2.5050×10^{-7}	2.5541×10^{-5}	—	—
	4	3.6087×10^{-7}	3.6973×10^{-5}	—	—
	5	4.9139×10^{-7}	5.0616×10^{-5}	—	—